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(*********************************
  BREAKTHROUGHS*************************************************************
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(* this file shows the differentiability of a worker's benefit from investment.

(* notation: la stands for lambda_h, pi for \pi, qs for \underline{p},
higha for Ua(h,qa,qb), lowa for Ua(l,qa,qb), highb for Ub(h,qa,qb),
lowb for Ub(l,qa,qb), t for how long it takes for belief to
 drop from qa to qb when the rate is lambda_h, ts for how long it
 takes for belief to drop from qb to qs when the rate is lambda_h/2.
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In[•]:=
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Quit[]
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(* step one: assuming that qa>qb, we write down the benefit from investment
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In[•]:=
```
higha = 1 - Exp[- la t] + Exp[- la t] (1 - Exp[- r t] + Exp[- r t] S[h, qb]);
lowa = 1 - Exp[- r t] + Exp[- r t] S[ell, qb];
highb = Exp[- r t] (1 - qa + qa Exp[- la t]) S[h, qb];
lowb = Exp[- r t] (1 - qa + qa Exp[- la t]) S[ell, qb];
F1[t1_] := 1 - Exp[- la t1 / 2]
F2[t2_] := qb (1 - Exp[- la t2 / 2])
f1[t1_] := D[F1[t1], t1]
f2[t2_] := D[F2[t2], t2]
S[h, qb] =
  Integrate[((e^(-r t1) + 1/2 (1 - e^(-r t1))) (1 - F2[t1]) + ∫_0^t1 1/2 (1 - e^(-r t2)) f2[t2] dt2) f1[t1],
    {t1, 0, ts}] + ((1/2 (1 - e^(-r ts))) (1 - F2[ts]) + ∫_0^ts 1/2 (1 - e^(-r t2)) f2[t2] dt2) (1 - F1[ts]);
S[ell, qb] = Integrate[1/2 (1 - e^(-r t2)) f2[t2], {t2, 0, ts}] + 1/2 (1 - e^(-r ts)) (1 - F2[ts]);
```

$$\left\{t \to \frac{\text{Log}\left[\frac{qa\ (1-qb)}{(1-qa)\ qb}\right]}{la},\ ts \to \frac{2\ \text{Log}\left[\frac{qb\ (1-qs)}{(1-qb)\ qs}\right]}{la}\right\}$$

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(* step two: the benefit from investment for arbitrary {qa,qb} are:
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In[]:=
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FullSimplify[higha - lowa /. {t → Log[(qa (1-qb))/((1-qa) qb)]/la, ts → 2 Log[(qb (1-qs))/((1-qb) qs)]/la}]
```

Out[]=

$$\left(\left(\frac{qa\,(-1+qb)}{(-1+qa)\,qb}\right)^{-\frac{r}{la}}\left(\frac{qb\,(-1+qs)}{(-1+qb)\,qs}\right)^{-\frac{2r}{la}}\left((-1+qs)^2\left(\frac{qb\,(-1+qs)}{(-1+qb)\,qs}\right)^{\frac{2r}{la}}\left(la\left(-qa+qb^2\right)+2\left(-qa+qb\right)r\right)-(-1+qb)^2\left(la\left(qa+\left(-2+qs\right)qs\right)+2\,(qa-qs)\,r\right)\right)\right/\left(2\,qa\left(-1+qb\right)(-1+qs)^2\left(la+2\,r\right)\right)$$

In[]:=
```
FullSimplify[highb - lowb /. {t → Log[(qa (1-qb))/((1-qa) qb)]/la, ts → 2 Log[(qb (1-qs))/((1-qb) qs)]/la} /. {qa → qb, qb → qa}]
```

Out[]=

$$\left(qb\left(\frac{(-1+qa)\,qb}{qa\left(-1+qb\right)}\right)^{-\frac{la+r}{la}}\left(\frac{qa\,(-1+qs)}{(-1+qa)\,qs}\right)^{-\frac{2r}{la}}\left(la\,qa\,(-1+qs)^2\left(\frac{qa\,(-1+qs)}{(-1+qa)\,qs}\right)^{\frac{2r}{la}}-(-1+qa)\left(la\left(qa+\left(-2+qs\right)qs\right)+2\,(qa-qs)\,r\right)\right)\right/\left(2\,qa^2\,(-1+qs)^2\left(la+2\,r\right)\right)$$

```
Ba[qa_, qb_] := Piecewise[
  {{pi (((((qa (-1+qb))/((-1+qa) qb))^(-r/la) ((qb (-1+qs))/((-1+qb) qs))^(-2r/la) ((-1+qs)^2 ((qb (-1+qs))/((-1+qb) qs))^(2r/la) (la (-qa+qb^2)+2 (-qa+qb) r)-(-1+qb)^2 (la (qa+(-2+qs) qs)+2 (qa-qs) r)))/(2 qa (-1+qb) (-1+qs)^2 (la+2 r))), qa > qb},
   {pi (((qb ((-1+qa) qb)/(qa (-1+qb)))^(-(la+r)/la) (qa (-1+qs))/((-1+qa) qs))^(-2r/la) ((la qa (-1+qs)^2 ((qa (-1+qs))/((-1+qa) qs))^(2r/la)-(-1+qa) (la (qa+(-2+qs) qs)+2 (qa-qs) r)))/(2 qa^2 (-1+qs)^2 (la+2 r))), qa ≤ qb}}]

(* step three:  we show that Ba is continuous and continuously differentiable.
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In[4]:=
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FullSimplify[
 FullSimplify[Ba[qa, qb], qa > qb] == FullSimplify[Ba[qa, qb], qa < qb] /. qa → qb]
```

Out[4]= True

In[5]:=
```
FullSimplify[FullSimplify[D[Ba[qa, qb], qa], qa > qb] ==
   FullSimplify[D[Ba[qa, qb], qa], qa < qb] /. qa → qb]
```

Out[5]= True

In[6]:=
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FullSimplify[FullSimplify[D[Ba[qa, qb], qb], qa > qb] ==
   FullSimplify[D[Ba[qa, qb], qb], qa < qb] /. qb → qa]
```

Out[6]= True