The Interval Structure of Optimal Disclosure

A Continuous-Type Example

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1 A continuous-type example

We next illustrate how to use theorem 3.1 and the techniques from the screening problem to solve for the optimal mechanism with a continuous-type example.

The type space is $\mathcal{T} = [0,1]$ and the state space is $\mathcal{S} = [-1,1]$. Receiver prefers to accept if and only if $s \geqslant 0$. We assume that $u(s)$ is constant over negative states, i.e., there exists $\eta > 0$ such that $u(s) = -\eta$ for every $s < 0$. Thus, the parameter η measures Receiver's loss when he accepts in a negative state. The density function of every type over states is also constant over negative states, i.e., there exists some $\underline{f} : \mathcal{T} \to \mathbf{R}_+$ such that $f(s,t) = \underline{f}(t)$ for every $s < 0$. Finally, we assume that the marginal distribution over types is uniform so that $\int_{-1}^{1} f(s,t)\,\mathrm{d}s = 1$ for every t. The last assumption is for convenience only.

The highest type's utility if he accepts under his prior belief is

$$\int_{-1}^{1} f(s,1)u(s)\,\mathrm{d}s = -\eta\underline{f}(t) + \int_{0}^{1} f(s,1)u(s)\mathrm{d}s.$$

We make the following simplifying assumption:

Assumption 1. *The highest type rejects in the absence of further information.*

Let $\overline{U}^*(\overline{q},t) = \int_0^{\overline{q}} f(s,t)u(s)\,\mathrm{d}s/\underline{f}(t)$ be the utility of type t from accepting on positive states $[0,\overline{q}]$, normalized by $\underline{f}(t)$. Type t's (normalized) utility from accepting on $\left[\underline{q},\overline{q}\right]$ is

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$\overline{U}^*(\overline{q}, t) + \eta \underline{q}$. We can view type t as a buyer with a quasilinear utility function who gets utility $\overline{U}^*(\overline{q}, t) + \eta \underline{q}$ from consuming quality $\overline{q}$ and paying price $-\eta \underline{q}$. The Spence-Mirrlees sorting condition $\frac{\partial}{\partial \overline{q}, \partial t} \overline{U}^*(\overline{q}, t) \geqslant 0$ holds in our setup, given the i.m.l. ratio assumption. From the theory of optimal screening we therefore know that the IC constraints hold if and only if $\overline{\pi}$ is monotone-increasing and

$$-\eta \underline{\pi}(t) = \overline{U}^*(\overline{\pi}(t), t) - \int_0^t \overline{U}_2^*(\overline{\pi}(\tau), \tau)\, \mathrm{d}\tau, \tag{1}$$

where $\overline{U}_2^*$ is the derivative of $\overline{U}^*$ with respect to the type. Note that given $\overline{\pi}(t)$, the IC constraints determine the "price" $-\eta \underline{\pi}(t)$ up to a constant, so we could add some $C \leqslant 0$ to the right-hand side; however, as in the monopoly screening problem, it is optimal to choose $C = 0$. Also, our setup has an additional requirement that $\underline{\pi}(t) \geqslant -1$. (In terms of the monopoly screening problem, this would correspond to an upper bound on the price.) Assumption 1 makes sure that this constraint does not bind.

For every $\overline{q} > 0$, we let $\overline{F}(\overline{q}, t) = \int_0^{\overline{q}} f(s, t)\, \mathrm{d}s$ be Sender's payoff if Receiver accepts in $[0, \overline{q}]$. Sender's payoff is given by

$$\begin{aligned} \int_0^1 \int_{\underline{\pi}(t)}^{\overline{\pi}(t)} f(s,t) \mathrm{d}s \mathrm{d}t &= \int_0^1 \left(-\underline{f}(t) \underline{\pi}(t) + \int_0^{\overline{\pi}(t)} f(s,t) \mathrm{d}s \right) \mathrm{d}t \\ &= \int_0^1 \left(\frac{\overline{U}^*(\overline{\pi}(t), t) \underline{f}(t) - \overline{U}_2^*(\overline{\pi}(t), t) \int_t^1 \underline{f}(\tau) \mathrm{d}\tau}{\eta} + \overline{F}(\overline{\pi}(t), t) \right) \mathrm{d}t. \end{aligned} \tag{2}$$

Thus, the Sender's problem in our setup has the same structure as the standard monopoly screening problem with quasilinear utility: we maximize (2) over all monotone-increasing functions $\overline{\pi} : [0, 1] \to [0, 1]$. As in the case of the monopoly screening problem, this is in general a control problem. Under some regularity assumptions, the monotonicity constraints do not bind, so we can continue without resorting to the control theory. Lastly, we fully characterize the optimal disclosure for a class of distributions, provide the necessary and sufficient condition for pooling to be optimal, and illustrate how the optimal mechanism is typically implemented with a continuous-type space.

We now assume that the joint density is

$$f(s, t) = \begin{cases} \frac{2}{\phi(2t-1)+4}, & \text{if } s \in [-1, 0), \\ \frac{2}{\phi(2t-1)+4} (\phi s(2t-1) + 1), & \text{if } s \in [0, 1]. \end{cases}$$

The parameter $\phi \in [0,1]$ measures how informative Receiver's type is: when $\phi = 0$, Receiver has no private information. As ϕ increases, a higher type becomes progressively more optimistic than a lower type does.

Sender's payoff and type t's utility if type t accepts on $[0,\overline{q}]$ are given, respectively, by

$$\overline{F}\left(\overline{q},t\right) = \frac{\overline{q}\left(\phi\overline{q}(2t-1)+2\right)}{\phi(2t-1)+4}, \text{ and } \overline{U}^*\left(\overline{q},t\right) = \frac{1}{6}\overline{q}^2\left(2\phi\overline{q}(2t-1)+3\right).$$

Substituting into the Sender's problem (2) and simplifying, we can write Sender's payoff as

$$\int_0^1 \left\{ \frac{2\left(\frac{\phi(2t-1)}{\phi(2t-1)+4} - \log\left(\frac{\phi+4}{\phi(2t-1)+4}\right)\right)}{3\eta}\overline{\pi}(t)^3 + \frac{\eta\phi(2t-1)+1}{\eta(\phi(2t-1)+4)}\overline{\pi}(t)^2 + \frac{2}{\phi(2t-1)+4}\overline{\pi}(t) \right\} \mathrm{d}t. \tag{3}$$

Sender maximizes his payoff by choosing $\overline{\pi}(t)$ to maximize the integrand pointwise. If we ignore the constraint $\overline{\pi}(t) \leqslant 1$, the integrand is maximized at:

$$\overline{\pi}^*(t) := \frac{2\eta}{\eta\phi(1-2t) - 1 + \sqrt{(\eta\phi(1-2t)+1)^2 + 4\eta(\phi(2t-1)+4)\log\left(\frac{\phi+4}{\phi(2t-1)+4}\right)}}. \tag{4}$$

When Receiver's type is not informative enough, $\overline{\pi}^*(t)$ is greater than 1. The integrand is maximized at the highest state 1 for every t. In this case, the optimal mechanism is pooling.

Proposition 1.1. *There exists an increasing function $\Phi(\cdot)$ such that Sender pools all types if and only if $\phi \leqslant \Phi(\eta)$: Sender sets $\overline{\pi}(t)$ to be 1, and $\underline{\pi}(t)$ is a constant which is chosen such that the lowest type is indifferent.*

Figure 1 shows how the optimal mechanism varies as (ϕ, η) vary. Assumption 1 states that the highest type rejects without further information, i.e., $6\eta \geqslant 2\phi + 3$. This corresponds to the parameter region above the solid line. The dashed line corresponds to the function $\Phi^{-1}(\cdot)$. To the right of the dashed line, semi-separating is optimal; to the left, pooling is optimal.

When $\phi > \Phi(\eta)$, the maximizer $\overline{\pi}^*(t)$ is below 1 when Receiver has the lowest type 0. We show that $\overline{\pi}^*(t)$ increases to 1 at some type $\hat{t}$. Thus, the integrand in (3) is maximized at $\overline{\pi}^*(t)$ when $t \leqslant \hat{t}$ and at 1 when $t > \hat{t}$. Given this $\overline{\pi}(t)$, we can derive $\underline{\pi}(t)$ based on the incentive constraints (1). We show in lemma 1.1 (in section 1.1) that both $\overline{\pi}^*(t)$ and the corresponding $\underline{\pi}(t)$ increase in η for any fixed t.

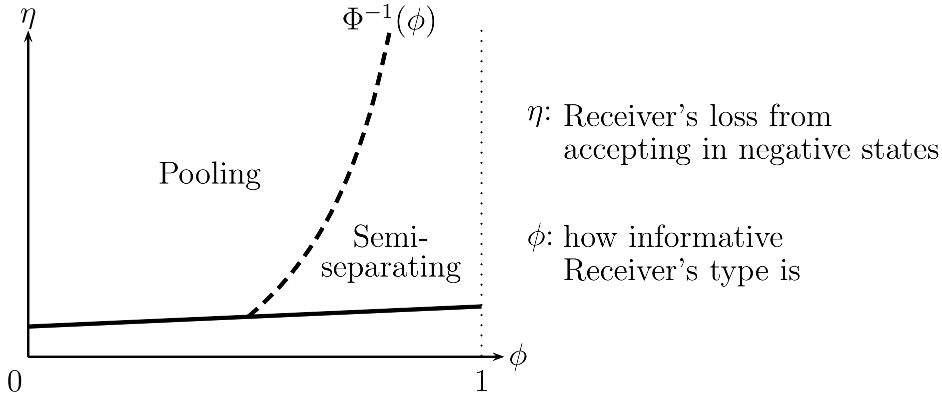

Figure 1: Pooling is optimal when ϕ is small.

Proposition 1.2. *Suppose that $\phi > \Phi(\eta)$. Then,*

(i) $\overline{\pi}(t) = \begin{cases} \overline{\pi}^*(t) & \text{for } t \leqslant \hat{t} \\ 1 & \text{for } t > \hat{t}, \end{cases}$ *where $\hat{t}$ is the critical type such that $\overline{\pi}^*\left(\hat{t}\right) = 1$;*

(ii) $\underline{\pi}(t)$ is determined by the incentive constraints (1).

Figure 2 illustrates the optimal mechanism in which Sender separates the lower types.[1] The left-hand side illustrates each type t's acceptance interval $[\underline{\pi}(t), \overline{\pi}(t)]$, which expands as t increases. The solid curves correspond to $\overline{\pi}(t)$ and $\underline{\pi}(t)$ for a lower η, and the dashed ones for a higher η. As η increases, Sender is less capable of persuading Receiver to accept in unfavorable states. Hence, both allocations $\overline{\pi}(t), \underline{\pi}(t)$ increase.

The right-hand side of figure 2 illustrates how to implement the optimal mechanism by examining the case in which η is 1. The solid curve corresponds to the cutoff function $z(s)$. For all the states below $\underline{\pi}(1)$, Sender recommends that all types reject. For any state s above $\underline{\pi}(1)$, Sender announces $z(s)$ and recommends that types above $z(s)$ accept. For a small segment of states surrounding state 0, $z(s)$ equals the lowest type, so all types accept. For any higher state s, Sender always mixes it with a lower state so that type $z(s)$ is indifferent when he is pronounced to be the cutoff type.

[1]The parameter values are $\phi = 1$, $\eta = 1$ for the solid line, and $\eta = 4/3$ for the dashed line.

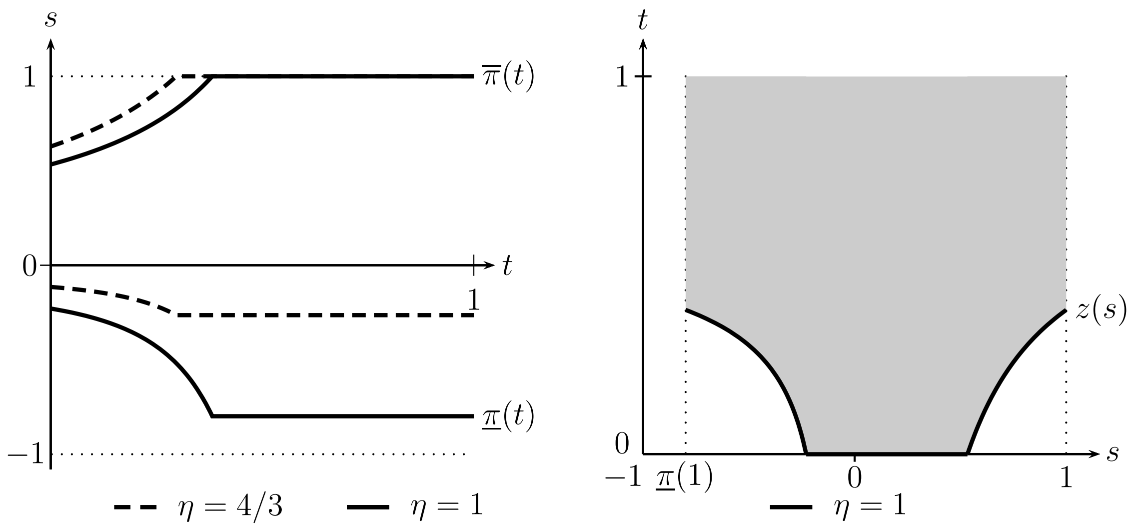

Figure 2: Optimal mechanism and its implementation

1.1 Proof of propositions 1.1 and 1.2

Proof. We let $G(\overline{\pi}(t))$ denote the integrand in (3). (To simplify exposition, the dependence of the integrand on t is omitted.) $G(\overline{\pi}(t))$ is a cubic function in $\overline{\pi}(t)$:

$$G(\overline{\pi}(t)) := b_1(t)\overline{\pi}(t)^3 + b_2(t)\overline{\pi}(t)^2 + b_3(t)\overline{\pi}(t),$$

where

$$b_1(t) := \frac{2\left(\frac{2\phi t-\phi}{2\phi t-\phi+4} - \log\left(\frac{\phi+4}{2\phi t-\phi+4}\right)\right)}{3\eta}, \quad b_2(t) := \frac{2\eta\phi t - \eta\phi + 1}{\eta(2\phi t - \phi + 4)}, \quad b_3(t) := \frac{2}{\phi(2t-1)+4}.$$

The first-order condition is $G'(\overline{\pi}(t)) = 3b_1(t)\overline{\pi}(t)^2 + 2b_2(t)\overline{\pi}(t) + b_3(t)$. It can be readily verified that $b_3(t) > 0$ for all $t \in [0,1]$. Moreover, $b_1(t)$ strictly increases in t; $b_1(0) < 0$; and $b_1(1) > 0$. We let $\tilde{t} \in (0,1)$ be the value which solves $b_1(t) = 0$.

When $t \geqslant \tilde{t}$, $G'(\overline{\pi}(t))$ is strictly positive for any $\overline{\pi}(t) \geqslant 0$, so $G(\overline{\pi}(t))$ is maximized at $\overline{\pi}(t) = 1$. When $t < \tilde{t}$, the equation $G'(\overline{\pi}(t)) = 0$ has a unique positive root since $b_1(t) < 0$ and $b_3(t) > 0$. This root is given by $\overline{\pi}^*(t)$ in (4). It can be easily verified that $\overline{\pi}^*(t)$ is monotone-increasing for $t \in \left[0, \tilde{t}\right]$ and goes to infinity as t approaches $\tilde{t}$.

If $\overline{\pi}^*(t)$ is above 1 at $t = 0$, then $G(\overline{\pi}(t))$ is maximized at $\overline{\pi}(t) = 1$ for every t. Thus, Sender pools all types when the following inequality holds:

$$\overline{\pi}^*(0) > 1 \longleftrightarrow \eta > \frac{(\phi - 4)\log\left(\frac{\phi+4}{4-\phi}\right)}{\phi - 1} - 1.$$

The right-hand side is monotone-increasing in ϕ, as desired.

If $\overline{\pi}^*(t)$ is below 1 at $t = 0$, there is a critical $\hat{t}$ which solves $\overline{\pi}(t) = 1$. Thus, $G(\overline{\pi}(t))$ is maximized at $\overline{\pi}^*(t)$ for $t \leqslant \hat{t}$ and at 1 for $t > \hat{t}$. □

Lemma 1.1. *Suppose that $\phi > \Phi(\eta)$. Both $\overline{\pi}(t)$ and $\underline{\pi}(t)$ increase pointwise in η. Thus, there exists $\bar{\eta}(\phi)$ such that the constraint $\underline{\pi}(t) \geqslant -1$ does not bind if and only if $\eta > \bar{\eta}(\phi)$.*

Proof of lemma 1.1. When t equals $1/2$, the value of $b_1(t)$, as defined in the proof of proposition 1.1, is $\frac{2\log\left(\frac{4}{\phi+4}\right)}{3\eta}$, which is negative given that $\eta > 0$ and $\phi > 0$. When t equals $1/2$, the value of (4) equals:

$$\frac{2\eta}{\sqrt{16\eta\log\left(\frac{\phi+4}{4}\right) + 1} - 1},$$

which is greater than 1. This implies that $\hat{t} < 1/2$. For the rest of the proof, we focus on the domain that $t \in [0, 1/2)$.

We first show that $\overline{\pi}(t)$ increases in η by showing that $1/\overline{\pi}(t)$ decreases in η. The derivative of $1/\overline{\pi}(t)$ w.r.t. η is negative if and only if

$$\begin{aligned}&\eta(\phi - 2\phi t) + 2\eta(\phi(2t-1) + 4)\log\left(\frac{\phi+4}{2\phi t - \phi + 4}\right) + 1\\ &\geqslant \sqrt{(-2\eta\phi t + \eta\phi + 1)^2 + 4\eta(2\phi t - \phi + 4)\log\left(\frac{\phi+4}{2\phi t - \phi + 4}\right)}.\end{aligned}$$

The left-hand side is concave in t, and it decreases in t at $t = 0$. Hence, the left-hand side decreases in t. Moreover, it is positive when $t = 1/2$, so the left-hand side is positive. Taking the power of both sides, we show that the inequality above holds.

We next show that $\underline{\pi}(0)$ increases in η. We solve for $\underline{\pi}(0)$ based on the condition that type 0's expected utility is zero:

$$\underline{\pi}(0) = \frac{2\eta\left(\eta\phi - 3\sqrt{(\eta\phi+1)^2 - 4\eta(\phi-4)\log\left(\frac{\phi+4}{4-\phi}\right)} + 3\right)}{3\left(\eta\phi + \sqrt{(\eta\phi+1)^2 - 4\eta(\phi-4)\log\left(\frac{\phi+4}{4-\phi}\right)} - 1\right)^3}.$$

It is easy to show that this term increases in η.

Lastly, we want to show that $\underline{\pi}(t)$ increases in η. To do so, we write $\underline{\pi}(t)$ (as well as $\overline{\pi}(t)$) as a function of t and η:

$$\underline{\pi}(t,\eta) = \frac{6\int_0^t \frac{2}{3}\phi\overline{\pi}(\tau,\eta)^3\,d\tau + \overline{\pi}(t,\eta)^2(\phi(2-4t)\overline{\pi}(t,\eta)-3)}{6\eta}.$$

We have shown above that $\underline{\pi}^{(0,1)}(0,\eta)$ is positive. Next, we show that $\underline{\pi}^{(1,1)}(t,\eta)$ is positive, that is, $\underline{\pi}^{(0,1)}(t,\eta)$ increases in t. This completes the proof that $\underline{\pi}^{(0,1)}(t,\eta)$ is positive, so $\underline{\pi}(t,\eta)$ increases in η.

We let $x(t,\eta)$ denote the square root term in $\overline{\pi}(t)$:

$$x(t,\eta) := \sqrt{(2\eta\phi(1-2t)+1)^2 + 8\eta(\phi(1-2t)-4)\log\left(\frac{2\phi t-\phi+4}{\phi+4}\right)}.$$

It can be easily verified that $x(t,\eta) - \eta\phi(2t-1) - 1 > 0$. Given this condition and the fact that $t \in (0, 1/2)$, $\phi \in (0,1)$, and $\eta > 0$, it follows that the derivative $\underline{\pi}^{(1,1)}(t,\eta)$ is positive if

$$a_1 x(t,\eta)^3 + a_2 x(t,\eta)^2 + a_3 x(t,\eta) + a_4 > 0, \tag{5}$$

where

$$\begin{aligned}
a_1 &= \eta(\phi(6t-3)-8)-3,\\
a_2 &= (\eta(\phi-2\phi t)+1)(\eta(7\phi(2t-1)+24)+1),\\
a_3 &= -(5\eta\phi(1-2t)+3)(\eta(\phi(2t-1)(\eta(\phi(2t-1)+8)-4)-24)-1),\\
a_4 &= -(\eta\phi(2t-1)-1)(\eta\phi(2t-1)+1)(\eta(\phi(2t-1)(\eta(\phi(2t-1)+8)-4)-24)-1).
\end{aligned}$$

It is easy to show that $a_2 > 0$ and that $x(t,\eta) > 1$. Moreover, the cubic inequality (5) is satisfied when $x(t,\eta) = 1$. We next rewrite the left-hand side of (5) as a quadratic function of $x(t,\eta)$:

$$a_2 x(t,\eta)^2 + (a_1 x(t,\eta)^2 + a_3)x(t,\eta) + a_4.$$

If we can show that $(a_1 x(t,\eta)^2 + a_3)$ is positive, then the quadratic function increases in $x(t,\eta)$ for any $x(t,\eta) \geqslant 1$. Given that the quadratic function is positive when $x(t,\eta) = 1$, the inequality (5) is satisfied. Next, we prove that $(a_1 x(t,\eta)^2 + a_3)$ is positive.

Substituting $x(t,\eta)$ into $(a_1 x(t,\eta)^2 + a_3)$, we find that this term is positive if

$$-\frac{2(\eta\phi(2t-1)(\eta\phi(2t-1)-4)+2)}{\eta(\phi(6t-3)-8)-3} + \log\left(\frac{2\phi t - \phi + 4}{\phi + 4}\right) > 0. \tag{6}$$

It is easy to verify that the left-hand side of (6) is convex in η. The inequality (6) holds when η equals zero. We are interested in the parameter region when $\overline{\pi}(t) < 1$. For fixed ϕ and t, $\overline{\pi}(t)$ is smaller than 1 if

$$\eta < \overline{\eta} := \frac{(\phi(2t-1)+4)\log\left(\frac{\phi+4}{2\phi t-\phi+4}\right)}{\phi(2t-1)+1} - 1.$$

When we substitute $\eta = \overline{\eta}$ into (6), the inequality is satisfied for any $t \in (0, 1/2)$ and $\phi \in (0, 1)$.

The left-hand side of (6) increases in η if and only if

$$\phi^2(1-2t)^2(\phi(6t-3)-8)\eta^2 - 6\phi^2(1-2t)^2\eta + 2(\phi(6t-3)+8) < 0.$$

The quadratic function on the left-hand side is concave and admits one positive root and one negative root. The positive root is given by

$$\tilde{\eta} := \frac{\sqrt{128 - 9\phi^2(1-2t)^2} + \phi(6t-3)}{\phi(2t-1)(\phi(6t-3)-8)}.$$

Given that $\eta > 0$, we obtain that the left-hand side of (6) decreases in η when $\eta < \tilde{\eta}$, and increases in η when $\eta \geqslant \tilde{\eta}$. We now have to discuss two cases, depending on whether $\tilde{\eta}$ is above or below $\overline{\eta}$:

1. If $\tilde{\eta} \geqslant \overline{\eta}$, the left-hand side of (6) decreases in η for any $\eta \in (0, \overline{\eta})$. We have shown that the left-hand side of (6) is positive at $\eta = 0$ and $\eta = \overline{\eta}$. Hence, the left-hand side of (6) is positive for any $\eta \in (0, \overline{\eta})$.

2. If $\tilde{\eta} \in (0, \overline{\eta})$, then the minimum of the left-hand side of (6) is achieved when $\eta = \tilde{\eta}$. We need to show that (6) holds when $\eta = \tilde{\eta}$. The condition $\tilde{\eta} < \overline{\eta}$ holds only if $\phi > 11/20$ and $t < 1/4$. When we substitute $\eta = \tilde{\eta}$ into (6), the inequality (6) is satisfied when we restrict attention to the parameter region $\phi > 11/20$ and $t < 1/4$.

By combining the two cases above, we have shown that (6) holds. □